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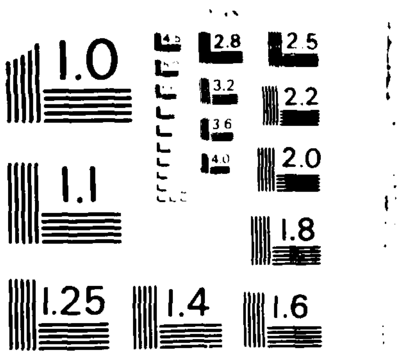

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19 ABSTRACT (Continue on reverse if necessary and identify by block number)

Wave-mode coordinates and scattering matrices are discussed in conjunction with the dynamic and wave propagation analyses of large space structures. Simple one-dimensional examples are given to illustrate how wave-mode coordinates and scattering matrices may be used to describe dynamics and wave propagation in such structures.

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INTRODUCTION

Recent proposals for orbiting space systems have stimulated research in the design, analysis and control of large space structures. Such structures are envisioned to be very large (one review paper [1] speculates about "a structure nearly the size of Manhattan Island") and are often in the form of lattice structures. A lattice structure is a network of slender elements which are connected by joints. This paper is concerned with the analysis of the dynamics and, in particular, the wave-like motion of lattice structures.

It is convenient to use matrix methods to analyze the dynamics of lattice structures. In particular, the dynamics of a lattice member may be described by a transfer matrix relationship, and the dynamics of a joint in a lattice structure may be described by a joint coupling matrix relationship. In order to study wave propagation in lattice structures, it is convenient to introduce wave-mode coordinates and scattering matrices. Wave-mode coordinates are obtained through the process of diagonalizing or uncoupling the transfer matrix relationship. When the transformation used to uncouple the transfer matrix is applied to the joint coupling matrix relationship, the scattering matrix is obtained.

In this paper, the derivations leading to wave-mode coordinates and scattering matrices in lattice structures are reviewed. In addition, simple one-dimensional examples are given to illustrate how wave-mode coordinates and scattering matrices may be used to describe dynamics and wave propagation in large space lattice structures.

ANALYSIS

State Vectors and Transfer Matrices

In the matrix formulation of the dynamics of lattice structures, the state or configuration of the lattice is determined by state vectors which exist at each point in the lattice. The state vector at a particular point is a column matrix whose components may be displacements, rotations, internal forces and internal moments. The actual components contained in a state vector at a particular point depend on the model used to describe the latt:ce at that point.

If the members of a lattice structure are slender, they may be modeled as one-dimensional continua. The components of the state vectors (and the state vectors themselves) of a one-dimensional continuous member are functions of time and one local spatial coordinate, say, x which extends along the length of the member. At a particular location $x = x_1$, the Fourier transform of the state vector of a one-dimensional member is given by [2]

$$\underline{\bar{z}}(x_1,\omega) = \int_{-\infty}^{\infty} \underline{z}(x_1,t)e^{-i\omega t}\, dt \tag{1}$$

where $\underline{z}(x_1,t)$ is the state vector at $x = x_1$, $\underline{\bar{z}}(x_1,\omega)$ is the Fourier transform of the state vector at $x = x_1$, t is time, ω is radian frequency, and $i = \sqrt{-1}$. The state vector $\underline{z}(x_1,t)$ is given in terms of the transformed state vector $\underline{\bar{z}}(x_1,\omega)$ by the inverse Fourier transform relationship [2]

$$\underline{z}(x_1,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{\bar{z}}(x_1,\omega)e^{i\omega t}\, d\omega \tag{2}$$

Throughout this paper, an overbar will signify a Fourier transform, and an underbar will signify a vector or a matrix.

The Fourier transforms of the state vectors at two locations $x = x_1$ and $x = x_2$ of a one-dimensional lattice member can be related by a transfer matrix according to

$$\underline{\bar{z}}(x_2,\omega) = \underline{T}(x_2 - x_1,\omega)\underline{\bar{z}}(x_1,\omega) \tag{3}$$

where $\underline{T}(x_2 - x_1,\omega)$ is the n x n transfer matrix (here n is the (even) number of components of the transformed state vectors $\underline{\bar{z}}(x_2,\omega)$ and $\underline{\bar{z}}(x_1,\omega)$), which is a function of the separation $(x_2 - x_1)$ and the frequency ω. The transfer matrix is derived from the equations of motion and the constitutive equations of the member. Transfer matrices for several specific types of lattice members are given in [3].

Wave-Mode Coordinates and Propagation Constants

The transfer matrix $\underline{T}$ in eqn. (3) generally contains nonzero elements off its main diagonal, and therefore the scalar equations relating the components of $\underline{\bar{z}}(x_2,\omega)$ and $\underline{\bar{z}}(x_1,\omega)$ are coupled. In order to study wave propagation in lattice structures, it is convenient to cast the transfer matrix relationship given by eqn. (3) into a different form. In particular, it is desirable to diagonalize the transfer matrix, and thus produce an uncoupled set of scalar equations. This diagonalization may be accomplished by the following procedure.

First, the eigenvectors of $\underline{T}$ are found, and the n x n wave-mode matrix $\underline{Y}(\omega)$ is assembled. The columns of $\underline{Y}(\omega)$ are the eigenvectors of the transfer matrix $\underline{T}$. Next, a new vector $\underline{\bar{w}}(x,\omega)$ is defined by the equation

$$\underline{\bar{w}}(x,\omega) = \underline{Y}^{-1}(\omega)\ \underline{\bar{z}}(x,\omega) \tag{4}$$

The vector $\underline{\bar{w}}(x,\omega)$ will be discussed and interpreted shortly. From eqn. (4) it follows that

$$\underline{\bar{z}}(x,\omega) = \underline{Y}(\omega)\underline{\bar{w}}(x,\omega) \tag{5}$$

Note that the existence of $\underline{Y}^{-1}(\omega)$ in eqn. (4) depends on the linear independence of the eigenvectors of $\underline{T}$ [4]. Substitution of eqn. (5) into eqn. (3) gives

$$\underline{Y}(\omega)\underline{\bar{w}}(x_2,\omega) = \underline{T}(x_2 - x_1,\omega)\underline{Y}(\omega)\underline{\bar{w}}(x_1,\omega) \tag{6}$$

Premultiplication of both sides of eqn. (6) by $\underline{Y}^{-1}(\omega)$ gives

$$\underline{\bar{w}}(x_2,\omega) = \underline{W}(\omega)\underline{\bar{w}}(x_1,\omega) \tag{7}$$

where

$$\underline{W}(\omega) = \underline{Y}^{-1}(\omega)\underline{T}(x_2 - x_1,\omega)\underline{Y}(\omega) \tag{8}$$

The n x n matrix $\underline{W}(\omega)$ is called the wave-mode propagation matrix. Assuming that the eigenvectors of $\underline{T}$ are linearly independent (and that, therefore, $\underline{Y}^{-1}(\omega)$ exists), the matrix $\underline{W}(\omega)$ is a diagonal matrix of the form [4]

$$\underline{W}(\omega) = \begin{bmatrix} K_1 & 0 & . & . & . & 0 \\ 0 & K_2 & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & 0 \\ 0 & . & . & . & 0 & K_n \end{bmatrix} \tag{9}$$

where the K_i ($i = 1,2, \ldots, n$) are the n eigenvalues of $\underline{T}$. For uniform lattice members, which are considered exclusively in this paper, the transfer matrix $\underline{T}$ is cross-symmetric, and therefore the eigenvalues of $\underline{T}$ occur in pairs (K_j, $1/K_j$), $j = 1,2, \ldots, n/2$ [5].

The vector $\bar{\underline{w}}(x, \omega)$ given by eqn. (4) is called the wave-mode vector, and the components of $\bar{\underline{w}}(x, \omega)$ are called wave-mode coordinates. The wave-mode coordinates are given in terms of the physical components of the state vector by eqn. (4), and the physical components of the state vector are given in terms of the wave-mode coordinates by eqn. (5). In general, each wave-mode coordinate will be a combination of all the components of the state vector. The wave-mode matrix $\underline{Y}(\omega)$, which gives the transformation between the wave-mode vector and the state vector, is similar in some respects to the modal matrix used in the analysis of multiple degree of freedom vibratory systems.

Because the matrix $\underline{W}(\omega)$ in eqn. (7) is diagonal, the wave-mode coordinates are uncoupled from each other as they propagate from $x = x_1$ to $x = x_2$ along the lattice member. The components of $\bar{\underline{w}}(x, \omega)$ are called wave-mode coordinates because they represent, for many lattice member models, propagating waves. Eqn. (9) can be written in the form

$$\underline{W}(\omega) = \begin{bmatrix} e^{\gamma_1(x_2-x_1)} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & e^{\gamma_2(x_2-x_1)} & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & e^{\gamma_n(x_2-x_1)} \end{bmatrix} \tag{10}$$

where the quantities γ_i $(i = 1,2, \ldots, n)$ satisfy the equations

$$K_i = e^{\gamma_i(x_2-x_1)} \tag{11}$$

or

$$\gamma_i = \frac{1}{(x_2-x_1)} \ln(K_i) \tag{12}$$

The quantities γ_i are called "propagation constants", and are in general complex. Since the eigenvalues K_i occur in pairs $(K_j, 1/K_j)$, the propagation constants occur in pairs $(\gamma_j, -\gamma_j)$, $j = 1,2, \ldots, n/2$. The general form of the propagation constant pair $(\gamma_j, -\gamma_j)$ is [6]

$$\gamma_j = \alpha(\omega) + ik(\omega) \tag{13}$$

$$-\gamma_j = -\alpha(\omega) - ik(\omega) \tag{14}$$

where $\alpha(\omega)$ is called the attenuation and $k(\omega)$ is called the wave number.

Propagating waves represented by wave-mode coordinates may be classified according to the nature of their corresponding propagation constants. If a propagation constant is purely imaginary, the wave represented by the corresponding wave-mode coordinate is not attenuated as it propagates. If a propagation constant has a nonzero real part and a nonzero imaginary part, the wave represented by the corresponding wave-mode coordinate is attenuated as it propagates. A propagation constant which is purely real corresponds to a wave-mode coordinate which does not represent a wave at all, but rather represents a nonpropagating spatially attenuated vibration. If the imaginary part of a propagation constant is linearly proportional to frequency ω, the wave represented by the corresponding wave-mode coordinate is nondispersive, and the phase velocity of the wave is given by the inverse of the constant of proportionality. If the imaginary part of a propagation constant is not linearly proportional to frequency, the wave represented by the corresponding wave-mode coordinate is dispersive.

The physical interpretation of the fact that propagation constants for uniform members occur in pairs $(\gamma_j, -\gamma_j)$ is that for uniform members identical waves may propagate in either the direction of increasing x or the direction of decreasing x. Propagation constants with a nonpositive real part and a negative imaginary part correspond to wave-mode coordinates which represent waves which propagate in the direction of increasing x, and propagation constants with a nonnegative real part and a positive imaginary part correspond to wave-mode coordinates which represent waves which propagate in the direction of decreasing x. For passive lattice members (that is, for lattice members with no external energy inputs), propagation constants with a positive imaginary part must also have a nonnegative real part, and propagation constants with a negative imaginary part must also have a nonpositive real part [6].

Joint Coupling Matrices and Scattering Matrices

The dynamics of a joint in a lattice structure may be described by a joint coupling relationship of the form [7]

$$\underline{B} \begin{Bmatrix} \bar{\underline{z}}_1 \\ \bar{\underline{z}}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{\underline{z}}_m \end{Bmatrix} = \bar{\underline{f}}_{ext} \tag{15}$$

where the $\bar{\underline{z}}_i$ (i = 1,2, ..., m) are the state vectors of the m members which are connected to the joint at the respective points of contact between the members and the joint, $\bar{\underline{f}}_{ext}$ is a vector containing the Fourier transforms of the external forces and moments which are applied to the joint, and $\underline{B}$ is a matrix called the joint coupling matrix. The joint coupling matrix is derived from the equations of motion of the joint, the constitutive equations of the joint, and the geometric compatibility requirements at the joint. The explicit form of the joint coupling matrix for arbitrary two and three-dimensional rigid joints with mass is given in [7].

Eqn. (15) is written in terms of the state vectors $\bar{\underline{z}}_i$ (i = 1,2, ..., m). In order to describe the dynamics of a joint in terms of wave-mode coordinates, the following transformations are introduced:

$$\bar{\underline{z}}_i = \underline{Y}_i(\omega)\bar{\underline{w}}_i, \quad i = 1,2, \ldots, m \tag{16}$$

where $\underline{Y}_i(\omega)$ is the wave-mode matrix of the i^{th} member which is attached to the joint, and $\bar{\underline{w}}_i$ is the wave-mode vector of the i^{th} member at the point of contact between the i^{th} member and the joint. Substitution of eqn. (16) into eqn. (15) gives

$$\underline{B} \begin{Bmatrix} \underline{Y}_1(\omega)\bar{\underline{w}}_1 \\ \underline{Y}_2(\omega)\bar{\underline{w}}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{Y}_m(\omega)\bar{\underline{w}}_m \end{Bmatrix} = \bar{\underline{f}}_{ext} \tag{17}$$

which can be written in the form

$$\underline{C} \begin{Bmatrix} \bar{\underline{w}}_1 \\ \bar{\underline{w}}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{\underline{w}}_m \end{Bmatrix} = \bar{\underline{f}}_{ext} \tag{18}$$

where

$$\underline{C} = \underline{B} \begin{bmatrix} \underline{Y}_1(\omega) & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \underline{Y}_2(\omega) & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ & & & & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & \underline{Y}_m(\omega) \end{bmatrix} \tag{19}$$

Eqn. (18) describes the dynamics of a joint in terms of wave-mode vectors $\underline{w}_i$. As discussed above, the wave-mode vectors contain wave-mode coordinates which represent waves travelling along the members. In describing the dynamics of a joint in terms of wave-mode coordinates, it is convenient to group the wave-mode coordinates into two groups: wave-mode coordinate which represent waves "entering" the joint, and wave-mode coordinates which represent waves "leaving" the joint. A wave-mode coordinate in the i^{th} member attached to the joint represents a wave entering the joint if

$$\mathrm{sgn}\left[\mathrm{Im}(\gamma) \cdot (\underline{n}_i \cdot \underline{e}_{xi})\right] = 1 \tag{20}$$

where γ is the propagation constant corresponding to the particular wave-mode coordinate, $\underline{e}_{xi}$ is a unit vector pointing in the direction of increasing x_i, where x_i is the (local) x-axis of the i^{th} member attached to the joint, and $\underline{n}_i$ is the outward unit normal to the joint at the point where the i^{th} member is attached to the joint. Eqn. (20) will be satisfied if the x_i-axis points "into" the joint and the wave-mode coordinate propagates in the direction of increasing x_i, or if the x_i-axis points "away from" the joint and the wave-mode coordinate propagates in the direction of decreasing x_i. A wave-mode coordinate in the i^{th} member attached to the joint represents a wave leaving the joint if

$$\mathrm{sgn}\left[\mathrm{Im}(\gamma) \cdot (\underline{n}_i \cdot \underline{e}_{xi})\right] = -1 \tag{21}$$

Eqn. (21) will be satisfied if the x_i-axis points "into" the joint and the wave-mode coordinate propagates in the direction of decreasing x_i, or if the x_i-axis points "away from" the joint and the wave-mode coordinate propagates in the direction of increasing x_i. Fig. 1 illustrates schematically wave-mode coordinates entering and leaving a generic joint.

The scalar equations in eqn. (18) may now be rearranged to give

$$\underline{D} \begin{Bmatrix} \bar{\underline{w}}_{out} \\ \\ \bar{\underline{w}}_{in} \end{Bmatrix} = \bar{\underline{f}}_{ext} \tag{22}$$

where $\bar{\underline{w}}_{out}$ is a vector containing all the wave-mode coordinates leaving the joint, $\bar{\underline{w}}_{in}$ is a vector containing all the wave-mode coordinates entering the joint, and $\underline{D}$ is the matrix obtained by rearranging the scalar equations of eqn. (18) into the form of eqn. (22). (The matrix manipulations leading to eqn. (22), and all the other matrix manipulations described in this section, will become clearer in the examples given below.) Eqn. (22) can be written as

$$\underline{D}_{out} \bar{\underline{w}}_{out} + \underline{D}_{in} \bar{\underline{w}}_{in} = \bar{\underline{f}}_{ext} \tag{23}$$

where $\underline{D}_{out}$ is the submatrix of $\underline{D}$ which multiplies $\bar{\underline{w}}_{out}$, and $\underline{D}_{in}$ is the submatrix of $\underline{D}$ which multiplies $\bar{\underline{w}}_{in}$. Premultiplication of both sides of eqn. (23) by $\underline{D}_{out}^{-1}$ (assuming that $\underline{D}_{out}^{-1}$ exists) gives

$$\bar{\underline{w}}_{out} + \underline{D}_{out}^{-1} \underline{D}_{in} \bar{\underline{w}}_{in} = \underline{D}_{out}^{-1} \bar{\underline{f}}_{ext} \tag{24}$$

or

$$\underline{\bar{w}}_{out} = \underline{S}(\omega)\underline{\bar{w}}_{in} + \underline{G}(\omega)\underline{\bar{f}}_{ext} \tag{25}$$

where

$$\underline{S}(\omega) = -\underline{D}_{out}^{-1} \underline{D}_{in} \tag{26}$$

and

$$\underline{G}(\omega) = \underline{D}_{out}^{-1} \tag{27}$$

The matrix $\underline{S}(\omega)$ is called the scattering matrix of the joint, and the matrix $\underline{G}(\omega)$ is called the wave-mode generation matrix of the joint.

Eqn. (25), which is merely a manipulated form of eqn. (17), is the final form of the description of the dynamics of a joint in terms of wave-mode coordinates. Eqn. (25) may be considered as an input-output relationship for the joint, and states that outgoing waves may be generated by incoming waves via the scattering matrix $\underline{S}(\omega)$, or may be generated by external forces (or moments) via the wave-mode generation matrix $\underline{G}(\omega)$. The scattering matrix $S(\omega)$ contains the transmission and reflection coefficients which the wave-mode coordinates encounter as they enter the joint. If the transmission and reflection coefficients in $\underline{S}(\omega)$ depend on the frequency ω, the scattering at the joint is called dispersive. If the transmission and reflection coefficients in $\underline{S}(\omega)$ are independent of frequency, the scattering is called nondispersive.

With a scattering matrix and a wave-mode generation matrix for each joint in a lattice, and a wave-mode matrix and a wave-mode propagation matrix for each member in a lattice, it is possible in principle to analyze wave propagation in lattice structures of arbitrary complexity. Following several intermediate examples, a complete simple example of such an analysis is given in the next section.

EXAMPLES

In this section, some simple one-dimensional examples are given to illustrate the concepts discussed above. Emphasis is placed on interpretation of the results, which in these simple examples can be checked by intuition and by elementary methods.

Example 1: Wave-Mode Coordinates for an Elastic Longitudinal Rod

In this example, wave-mode coordinates and propagation constants are derived for an elastic longitudinal rod. A section of an elastic longitudinal rod is shown in Fig. 2, which also shows the local coordinate x and the sign convention adopted for the displacements u_1 and u_2 and the forces F_1 and F_2. The rod is assumed to be uniform, with mass density ρ, cross-sectional area A, and elastic modulus E. The Fourier transform of the state vector at $x = x_1$ is

$$\bar{\underline{z}}_1 = \begin{Bmatrix} \bar{u}_1 \\ \bar{F}_1 \end{Bmatrix} \tag{28}$$

and the Fourier transform of the state vector at $x = x_2$ is

$$\bar{\underline{z}}_2 = \begin{Bmatrix} \bar{u}_2 \\ \bar{F}_2 \end{Bmatrix} \tag{29}$$

The vectors $\bar{\underline{z}}_1$ and $\bar{\underline{z}}_2$ are related by the transfer matrix relationship

$$\bar{\underline{z}}_2 = \underline{T}\ \bar{\underline{z}}_1 \tag{30}$$

where the transfer matrix $\underline{T}$ is given by [3]

$$\underline{T} = \begin{bmatrix} \cos\theta & \frac{\ell}{EA}\ \frac{\sin\theta}{\theta} \\ -\mu\ell\omega^2\ \frac{\sin\theta}{\theta} & \cos\theta \end{bmatrix}$$

where

$$\ell = x_2 - x_1 \tag{32}$$

$$\mu = \rho A \tag{33}$$

$$\theta = \ell\omega\sqrt{\frac{\rho}{E}} \tag{34}$$

The eigenvalues of the matrix $\underline{T}$ in eqn. (31) are [8]

$$K_1 = \cos\theta - i\sin\theta = e^{-i\theta} \tag{35}$$

$$K_2 = \cos\theta + i\sin\theta = e^{i\theta} \tag{36}$$

Therefore, the propagation constants given by eqn. (12) are

$$\gamma_1 = \frac{1}{\ell}\,(-i\theta) = -i\omega\sqrt{\frac{\rho}{E}} \tag{37}$$

$$\gamma_2 = \frac{1}{\ell}(i\theta) = i\omega\sqrt{\frac{\rho}{E}} \tag{38}$$

The eigenvectors of the matrix $\underline{T}$ in eqn. (31) are [8]

$$\underline{v}_1 = \begin{Bmatrix} 1 \\ -iR\omega \end{Bmatrix} \tag{39}$$

$$\underline{v}_2 = \begin{Bmatrix} 1 \\ iR\omega \end{Bmatrix} \tag{40}$$

where

$$R = A\sqrt{\rho E} \tag{41}$$

Note that the quantity R is the mechanical impedance of the rod [9].

Since the eigenvectors of the transfer matrix $\underline{T}$ are given by eqns. (39) and (40), the wave-mode matrix for the longitudinal rod is

$$\underline{Y}(\omega) = \begin{bmatrix} 1 & 1 \\ -iR\omega & iR\omega \end{bmatrix} \tag{42}$$

The inverse of the wave-mode matrix is

$$\underline{Y}^{-1}(\omega) = \frac{1}{2}\begin{bmatrix} 1 & -\frac{1}{iR\omega} \\ 1 & \frac{1}{iR\omega} \end{bmatrix} \tag{43}$$

The wave-mode propagation matrix $\underline{W}(\omega)$ given by eqn. (9) is

$$\underline{W}(\omega) = \underline{Y}^{-1}\underline{T}\,\underline{Y} = \begin{bmatrix} e^{-\gamma\ell} & 0 \\ 0 & e^{\gamma\ell} \end{bmatrix} \tag{44}$$

where

$$\gamma = i\omega\sqrt{\frac{\rho}{E}} \tag{45}$$

The wave-mode vector $\underline{\bar{w}}_2$ at $x = x_2$ and the wave-mode vector $\underline{\bar{w}}_1$ at $x = x_1$ are related according to eqn. (8) by

$$\underline{\bar{w}}_2 = \underline{W}(\omega)\underline{\bar{w}}_1 \tag{46}$$

Writing out eqn. (46) gives

$$\begin{Bmatrix} \bar{w}_2^+ \\ \bar{w}_2^- \end{Bmatrix} = \begin{bmatrix} e^{-\gamma\ell} & 0 \\ 0 & e^{\gamma\ell} \end{bmatrix} \begin{Bmatrix} \bar{w}_1^+ \\ \bar{w}_1^- \end{Bmatrix} \tag{47}$$

Eqn. (47) is the diagonalized form of the transfer matrix relationship given by eqn. (30). The components $\bar{w}_2^+$ and $\bar{w}_1^+$ of the wave-mode vectors $\underline{\bar{w}}_2$ and $\underline{\bar{w}}_1$ in eqn. (47) are written with a plus (+) superscript since the propagation constant $-\gamma$ which corresponds to $\bar{w}_2^+$ and $\bar{w}_1^+$ has a negative imaginary part, indicating that $\bar{w}_2^+$ and $\bar{w}_1^+$ represent waves which travel in the direction of increasing x. Similarly, the components $\bar{w}_2^-$ and $\bar{w}_1^-$ are written with a minus (-) superscript since the propagation constant γ which corresponds to $\bar{w}_2^-$ and $\bar{w}_1^-$ has a positive imaginary part, indicating that $\bar{w}_2^-$ and $\bar{w}_1^-$ represent waves which travel in the direction of decreasing x.

From eqns. (4) and (43), the wave-mode coordinates for the longitudinal rod are given in terms of the physical components of the state vector by

$$\begin{Bmatrix} \bar{w}^+(x,\omega) \\ \bar{w}^-(x,\omega) \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\frac{1}{iR\omega} \\ 1 & \frac{1}{iR\omega} \end{bmatrix} \begin{Bmatrix} \bar{u}(x,\omega) \\ \bar{F}(x,\omega) \end{Bmatrix} \tag{48}$$

which when written out gives the scalar equations

$$\bar{w}^+(x,\omega) = \frac{1}{2}\left(\bar{u}(x,\omega) - \frac{1}{iR\omega}\bar{F}(x,\omega)\right) \tag{49}$$

$$\bar{w}^-(x,\omega) = \frac{1}{2}\left(\bar{u}(x,\omega) + \frac{1}{iR\omega}\bar{F}(x,\omega)\right) \tag{50}$$

Note that each wave-mode coordinate contains information about both the displacement $\bar{u}$ and the force $\bar{F}$. From eqns. (5) and (40), the physical components of the state vector are given in terms of the wave-mode coordinates by

$$\begin{Bmatrix} \bar{u}(x,\omega) \\ \bar{F}(x,\omega) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ -iR\omega & iR\omega \end{bmatrix} \begin{Bmatrix} \bar{w}^+(x,\omega) \\ \bar{w}^-(x,\omega) \end{Bmatrix} \tag{51}$$

which when written out gives the scalar equations

$$\bar{u}(x,\omega) = \bar{w}^+(x,\omega) + \bar{w}^-(x,\omega) \tag{52}$$

$$\bar{F}(x,\omega) = -iR\omega\, \bar{w}^+(x,\omega) + iR\omega\, \bar{w}^-(x,\omega) \tag{53}$$

Eqn. (52) expresses the familiar fact that the displacement in an elastic longitudinal rod may be written as the sum of a wave travelling in the direction of increasing x and a wave travelling in the direction of decreasing x. Eqn. (53) states that the force (or stress) in the rod can also be written as the sum of the same two waves, scaled (in the frequency domain) by the appropriate factors.

The eigenvectors given by eqns. (39) and (40) remain eigenvectors when multiplied by any constant. By choosing the first components of the eigenvectors to be unity, the wave-mode coordinates in eqn. (48) are given the dimensions of displacement. If the second components of the eigenvectors are chosen to be unity, the resulting wave-mode coordinates will have the dimensions of force. Both sets of wave-mode coordinates will satisfy eqn. (47).

The propagation constants given by eqns. (37) and (38) are purely imaginary. Thus, waves on an elastic longitudinal rod are not attenuated as they propagate. Further, the imaginary parts of the propagation constants given by eqns. (37) and (38) are linearly proportional to frequency, and thus waves in an elastic rod are nondispersive. The phase velocity of the waves, which as stated above is equal to the inverse of the constant of proportionality, is, as is well known, $\sqrt{E/\rho}$.

Example 2: Joint Coupling Matrix for a Rigid One-Dimensional Joint with Mass

In this example, the joint coupling matrix for a one-dimensional rigid joint is derived. A rigid one-dimensional joint with mass m is shown in Fig. 3a Two members, denoted by 1 and 2, are attached to the joint. The joint is called one-dimensional because all members attached to the joint lie along a line, and the joint is constrained to move only along that line. The joint is subjected to an external force F_J. Fig. 3a also shows local coordinate axes x_1 and x_2 for members 1 and 2, respectively, and a joint coordinate axis x_J. Fig. 3b shows the components u_i and F_i $(i = 2,3)$ of the state vectors of the two members at their respective points of contact with the joint, and the components u_J and F_J of the state vector of the joint.

Since the joint is rigid, the geometric compatibility requirement at the joint is

$$u_2 = u_3 = u_J \tag{54}$$

Therefore,

$$\bar{u}_2 = \bar{u}_3 = \bar{u}_J \tag{55}$$

from which it follows that

$$-\bar{u}_2 + \bar{u}_3 = 0 \tag{56}$$

Applying Newton's Second Law to the joint gives, in terms of the Fourier transforms of the displacements and forces,

$$-\bar{F}_2 + \bar{F}_3 + \bar{F}_J = -m\omega^2\bar{u}_J \tag{57}$$

Substitution of eqn. (54) into eqn. (57) gives

$$-\bar{F}_2 + \bar{F}_3 + \bar{F}_J = -m\omega^2\bar{u}_2 \tag{58}$$

or

$$-m\omega^2\bar{u}_2 + \bar{F}_2 - \bar{F}_3 = \bar{F}_J \tag{59}$$

Eqns. (56) and (59) can be written as

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ -m\omega^2 & 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{F}_2 \\ \bar{u}_3 \\ \bar{F}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{F}_J \end{Bmatrix} \tag{60}$$

Eqn. (60) is in the form of the general joint coupling relationship given by eqn. (15), with

$$\underline{B} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -m\omega^2 & 1 & 0 & -1 \end{bmatrix} \tag{61}$$

Eqn. (61) gives the joint coupling matrix for the one-dimensional joint of Fig. 3.

This simple example illustrates the general procedure for deriving a joint coupling matrix. The joint coupling relationship for rigid joints consists essentially of geometric compatibility requirements and dynamic requirements, written in terms of Fourier transforms and in matrix form. In [7], the procedure used to derive the joint coupling matrix of eqn. (61) is applied to more complex two and three-dimensional rigid joints.

Example 3: Scattering Matrix for the Junction of Two Elastic Longitudinal Rods

Two elastic longitudinal rods connected by a rigid one-dimensional joint with mass m are shown in Fig. 4a. Rod 1 has mass density ρ_1, cross-sectional area A_1, and elastic modulus E_1. Rod 2 has mass density ρ_2, cross-sectional area A_2, and elastic modulus E_2.

As discussed in the previous example, the dynamics of the joint in Fig. 4a are described by the joint coupling relationship given by eqn. (60). In this example, the process of transforming eqn. (60) into a scattering relationship of the form of eqn. (25) is illustrated.

From eqn. (51), the state vectors at points 2 and 3 in Fig. 4 are related to the wave-mode coordinates at points 2 and 3 by

$$\begin{Bmatrix} \bar{u}_2 \\ \bar{F}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ -iR_1\omega & iR_1\omega \end{bmatrix} \begin{Bmatrix} \bar{w}_2^+ \\ \bar{w}_2^- \end{Bmatrix} \tag{62}$$

and

$$\begin{Bmatrix} \bar{u}_3 \\ \bar{F}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ -iR_2\omega & iR_2\omega \end{bmatrix} \begin{Bmatrix} \bar{w}_3^+ \\ \bar{w}_3^- \end{Bmatrix} \tag{63}$$

where

$$R_1 = A_1\sqrt{\rho_1 E_1} \tag{64}$$

$$R_2 = A_2\sqrt{\rho_2 E_2} \tag{65}$$

The waves represented by $\bar{w}_2^+$, $\bar{w}_2^-$, $\bar{w}_3^+$, and $\bar{w}_3^-$ are shown schematically by arrows in Fig. 4b. Substitution of eqns. (62) and (63) into eqn. (60) gives

$$\begin{bmatrix}-1 & 0 & 1 & 0\\ -m\omega^2 & 1 & 0 & -1\end{bmatrix}\left\{\begin{matrix}\begin{bmatrix}1 & 1\\ -iR_1\omega & iR_1\omega\end{bmatrix}\begin{Bmatrix}\bar{w}_2^+\\ \bar{w}_2^-\end{Bmatrix}\\ \begin{bmatrix}1 & 1\\ -iR_2\omega & iR_2\omega\end{bmatrix}\begin{Bmatrix}\bar{w}_3^+\\ \bar{w}_3^-\end{Bmatrix}\end{matrix}\right\} = \begin{Bmatrix}0\\ \bar{F}_J\end{Bmatrix} \qquad (66)$$

which can written as

$$\begin{bmatrix}-1 & 0 & 1 & 0\\ -m\omega^2 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 0 & 0\\ -iR_1\omega & iR_1\omega & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & -R_2\omega & iR_2\omega\end{bmatrix}\begin{Bmatrix}\bar{w}_2^+\\ \bar{w}_2^-\\ \bar{w}_3^+\\ \bar{w}_3^-\end{Bmatrix} = \begin{Bmatrix}0\\ \bar{F}_J\end{Bmatrix} \qquad (67)$$

Eqn. (67) is in the form of eqn. (18). Multiplying out eqn. (67) gives

$$\begin{bmatrix}-1 & -1 & 1 & 1\\ (-m\omega^2-iR_1\omega) & (-m\omega^2+iR_1\omega) & iR_2\omega & -iR_2\omega\end{bmatrix}\begin{Bmatrix}\bar{w}_2^+\\ \bar{w}_2^-\\ \bar{w}_3^+\\ \bar{w}_3^-\end{Bmatrix} = \begin{Bmatrix}0\\ \bar{F}_J\end{Bmatrix} \qquad (68)$$

Note from Fig. 4b that $\bar{w}_2^-$ and $\bar{w}_3^+$ represent waves leaving the joint, and $\bar{w}_2^+$ and $\bar{w}_3^-$ represent waves entering the joint. Rearranging the order of the scalar equations contained in eqn. (68) gives

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ (-m\omega^2+iR_1\omega) & iR_2\omega & (-m\omega^2-iR_1\omega) & -iR_2\omega \end{bmatrix} \begin{Bmatrix} \bar{w}_2^- \\ \bar{w}_3^+ \\ \bar{w}_2^+ \\ \bar{w}_3^- \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{F}_J \end{Bmatrix} \tag{69}$$

Eqn. (69) is in the form of eqn. (22) with

$$\underline{\bar{w}}_{out} = \begin{Bmatrix} \bar{w}_2^- \\ \bar{w}_3^+ \end{Bmatrix} \tag{70}$$

and

$$\underline{\bar{w}}_{in} = \begin{Bmatrix} \bar{w}_2^+ \\ \bar{w}_3^- \end{Bmatrix} \tag{71}$$

Eqn. (69) is eqivalent to

$$\begin{bmatrix} -1 & 1 \\ (-m\omega^2+iR_1\omega) & iR_2\omega \end{bmatrix} \begin{Bmatrix} \bar{w}_2^- \\ \bar{w}_3^+ \end{Bmatrix} + \begin{bmatrix} -1 & 1 \\ (-m\omega^2-iR_1\omega) & -iR_2\omega \end{bmatrix} \begin{Bmatrix} \bar{w}_2^+ \\ \bar{w}_3^- \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{F}_J \end{Bmatrix} \tag{72}$$

Eqn. (72) is in the form of eqn. (23), with

$$\underset{\sim}{D}_{out} = \begin{bmatrix} -1 & 1 \\ (-m\omega^2 + iR_1\omega) & iR_2\omega \end{bmatrix} \tag{73}$$

and

$$\underset{\sim}{D}_{in} = \begin{bmatrix} -1 & 1 \\ (-m\omega^2 - iR_1\omega) & -iR_2\omega \end{bmatrix} \tag{74}$$

The inverse of the matrix $\underset{\sim}{D}_{out}$ given by eqn. (73) is

$$\underset{\sim}{D}_{out}^{-1} = \begin{bmatrix} \dfrac{-iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega} & \dfrac{1}{-m\omega^2+ iR_1\omega+iR_2\omega} \\ \dfrac{-m\omega^2+iR_1\omega}{-m\omega^2+iR_1\omega+iR_2\omega} & \dfrac{1}{-m\omega^2+iR_1\omega+iR_2\omega} \end{bmatrix} \tag{75}$$

Premultiplication of both sides of eqn. (72) by $\underset{\sim}{D}_{out}^{-1}$ gives, after some manipulation,

$$\begin{Bmatrix} \bar{w}_2^- \\ \bar{w}_3^+ \end{Bmatrix} = \begin{bmatrix} \dfrac{m\omega^2+iR_1\omega-iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega} & \dfrac{2iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega} \\ \dfrac{2iR_1}{-m\omega^2+iR_1\omega+iR_2\omega} & \dfrac{m\omega^2-iR_1\omega+iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega} \end{bmatrix} \begin{Bmatrix} \bar{w}_2^+ \\ \bar{w}_3^- \end{Bmatrix}$$

$$+ \begin{bmatrix} \dfrac{-iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega} & \dfrac{1}{-m\omega^2+iR_1\omega+iR_2\omega} \\ \dfrac{-m\omega^2+iR_1\omega}{-m\omega^2+iR_1\omega+iR_2\omega} & \dfrac{1}{-m\omega^2+iR_1\omega+iR_2\omega} \end{bmatrix} \begin{Bmatrix} 0 \\ \bar{F}_J \end{Bmatrix} \tag{76}$$

Eqn. (76) is in the form of eqn. (25), with the scattering matrix $\underline{S}(\omega)$ given by

$$\underline{S}(\omega) = \begin{bmatrix} \dfrac{m\omega^2+iR_1\omega-iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega} & \dfrac{2iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega} \\ \dfrac{2iR_1\omega}{-m\omega^2+iR_1\omega+iR_2\omega} & \dfrac{m\omega^2-iR_1\omega+iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega} \end{bmatrix} \tag{77}$$

and the wave-mode generation matrix $\underline{G}(\omega)$ given by

$$\underline{G}(\omega) = \begin{bmatrix} \dfrac{-iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega} & \dfrac{1}{-m\omega^2+iR_1\omega+iR_2\omega} \\ \dfrac{-m\omega^2+iR_1\omega}{-m\omega^2+iR_1\omega+iR_2\omega} & \dfrac{1}{-m\omega^2+iR_1\omega+iR_2\omega} \end{bmatrix} \tag{78}$$

Eqn. (76) is the description of the dynamics of the joint of Fig. 4 in terms of wave-mode coordinates. Note that the description of the joint dynamics in terms of the physical components of the state vectors (eqn. (60)) contains only information about the dynamics of the joint. On the other hand, the description of the joint dynamics in terms of wave coordinates given by eqn. (76), which implicitly contains the joint coupling matrix given by eqn. (61) and the wave-mode matrices given by eqns. (62) and (63), contains information about both the joint dynamics and the dynamics of the members attached to the joint.

The quantity

$$\frac{m\omega^2+iR_1\omega-iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega}$$

in the scattering matrix $\underline{S}(\omega)$ given by eqn. (77) is the reflection coefficient for $\bar{w}_2^+$, which represents a wave incident upon the joint from the left, and the quantity

$$\frac{2iR_1\omega}{-m\omega^2+iR_1\omega+iR_2\omega}$$

is the transmission coefficient for $\bar{w}_2^+$. Similarly, the quantity

$$\frac{m\omega^2-iR_1\omega+iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega}$$

is the reflection coefficient for $\bar{w}_3^-$, which represents a wave incident upon the joint from the right, and the quantity

$$\frac{2iR_2\omega}{-m\omega^2+iR_1\omega+iR_2\omega}$$

is the transmission coefficient for $\bar{\bar{w}}_3$. Note that since each wave-mode coordinate contains information about both displacement and force (or stress), it is not necessary when using wave-mode coordinates to derive one set of reflection and transmission coefficients for displacement waves and a different set of reflection and transmission coefficients for force (or stress) waves, as is done, for example, in [9]. Since the wave-mode coordinates used here, which are defined by eqn. (48), have the dimensions of displacement, the reflection and transmission coefficients in the scattering matrix $\underline{S}(\omega)$ given by eqn. (77) correspond to the displacement reflection and transmission coefficients given in [9].

The transmission and reflection coefficients contained in the scattering matrix $\underline{S}(\omega)$ given by eqn. (77) depend on the frequency ω. Therefore, this is an example of dispersive scattering. As the frequency ω becomes very large, the scattering matrix $\underline{S}(\omega)$ given by eqn. (77) approaches

$$\underline{S}(\omega) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{79}$$

Thus, for very high frequencies, the scattering is nondispersive, and the reflection coefficients are equal to -1, while the transmission coefficients are equal to zero. Thus for very high frequencies, the joint in Fig. 4 behaves like a rigid boundary.

If the mass of the joint in Fig. 4 is zero, the scattering matrix $\underline{S}(\omega)$ given by eqn. (77) becomes

$$\underline{S}(\omega) = \begin{bmatrix} \dfrac{R_1 - R_2}{R_1 + R_2} & \dfrac{2R_2}{R_1 + R_2} \\ \\ \dfrac{2R_1}{R_1 + R_2} & \dfrac{R_2 - R_1}{R_1 + R_2} \end{bmatrix} \tag{80}$$

The scattering matrix given by eqn. (80) contains the commonly derived (see, for example, [9]) reflection and transmission coefficients for displacement waves at the junction of two collinear longitudinal rods.

Example 4: Analysis of a One-Dimensional Lattice Structure

In this fourth and final example, which utilizes the results of the previous three examples, the one-dimensional lattice of Fig. 5 is analyzed using wave-mode coordinates and scattering matrices. The lattice consists of two elastic longitudinal rods connected by a rigid one-dimensional joint. It is assumed that the joint is massless. Rod 1 has mass density ρ_1, cross-sectional area A_1, elastic modulus E_1, and length ℓ_1. Rod 2 has mass density ρ_2, cross-sectional area A_2, elastic modulus E_2, and length ℓ_2. The joint is subjected to a given force $\mathscr{F}(t)$, and it is desired to find the resulting force $F_1(t)$ at point 1.

The scattering matrix relationship at the joint is given by eqn. (76) as

$$\begin{Bmatrix} \bar{w}_2^- \\ \bar{w}_3^+ \end{Bmatrix} = \begin{bmatrix} r_1 & t_1 \\ t_2 & r_2 \end{bmatrix} \begin{Bmatrix} \bar{w}_2^+ \\ \bar{w}_3^- \end{Bmatrix} + \begin{Bmatrix} \dfrac{\bar{\bar{\mathscr{F}}}}{i\omega(R_1+R_2)} \\ \dfrac{\bar{\mathscr{F}}}{i\omega(R_1+R_2)} \end{Bmatrix} \tag{81}$$

where

$$r_1 = \frac{R_1 - R_2}{R_1 + R_2} \tag{82}$$

$$r_2 = \frac{R_2 - R_1}{R_1 + R_2} \tag{83}$$

$$t_1 = \frac{2R_2}{R_1 + R_2} \tag{84}$$

$$t_2 = \frac{2R_1}{R_1 + R_2} \tag{85}$$

and R_1 and R_2 are given by eqns. (64) and (65), respectively.

The components of the wave-mode vector $\underline{\bar{w}}_1$ at point 1 in Fig. 5 satisfy the relationship

$$(\bar{w}_1^+) = r_o(\bar{w}_1^-) \tag{86}$$

where r_o is the reflection coefficient at the left-hand boundary (point 1) of rod 1. The quantity r_o may be derived from the boundary conditions at point 1. If, for example, point 1 is a fixed end, $r_o = -1$, and if point 1 is a free end, $r_o = 1$ [10]. Similarly, the components of the wave-mode vector $\underline{\bar{w}}_4$ at point 4 in Fig. 5 satisfy the relationship

$$(\bar{w}_4^-) = r_3(\bar{w}_4^+) \tag{87}$$

where r_3 is the reflection coefficient at the right-hand boundary (point 4) of rod 2. Again, the quantity r_3 may be derived from the boundary condition at point 4; if point 4 is a fixed end, $r_3 = -1$, and if point 4 is a free end, $r_3 = 1$. The quantities r_o in eqn. (86) and r_3 in eqn. (87) may be considered as 1 x 1 scattering matrices. It is assumed that the quantities r_o and r_3 are independent of frequency. Eqns. (81), (86) and (87) are the scattering relationships for the lattice of Fig. 5.

The components of the wave-mode vector $\bar{\underline{w}}_2$ at point 2 in Fig. 5 and the components of the wave-mode vector $\bar{\underline{w}}_1$ at point 1 in Fig. 5 are related by a wave-mode propagation matrix according to eqn. (47) as

$$\begin{Bmatrix} \bar{w}_2^+ \\ \bar{w}_2^- \end{Bmatrix} = \begin{bmatrix} e^{-\gamma_1 \ell_1} & 0 \\ 0 & e^{\gamma_1 \ell_1} \end{bmatrix} \begin{Bmatrix} \bar{w}_1^+ \\ \bar{w}_1^- \end{Bmatrix} \tag{88}$$

where

$$\gamma_1 = i\omega \sqrt{\frac{\rho_1}{E_1}} \tag{89}$$

Similarly, the components of the wave-mode vector $\bar{\underline{w}}_4$ at point 4 in Fig. 5 and the components of the wave-mode vector $\bar{\underline{w}}_3$ at point 3 in Fig. 5 are related by

$$\begin{Bmatrix} \bar{w}_4^+ \\ \bar{w}_4^- \end{Bmatrix} = \begin{bmatrix} e^{-\gamma_2 \ell_2} & 0 \\ 0 & e^{\gamma_2 \ell_2} \end{bmatrix} \begin{Bmatrix} \bar{w}_3^+ \\ \bar{w}_3^- \end{Bmatrix} \tag{90}$$

where

$$\gamma_2 = i\omega\sqrt{\frac{\rho_2}{E_2}} \tag{91}$$

Eqns. (88) and (90) are the wave-mode propagation relationships for the lattice of Fig. 5.

Eqns. (81), (86), (87), (88) and (90) contain eight scalar equations for the eight unknowns $\bar{w}_1^+$, $\bar{w}_1^-$, $\bar{w}_2^+$, $\bar{w}_2^-$, $\bar{w}_3^+$, $\bar{w}_3^-$, $\bar{w}_4^+$ and $\bar{w}_4^-$. These equations may be solved for $\bar{w}_1^+$ and $\bar{w}_1^-$ to give [11]

$$(\bar{w}_1^+) = \frac{\bar{\mathcal{F}}}{i\omega(R_1+R_2)} \; \frac{r_o e^{-\gamma_1\ell_1}\left[1 + (t_1r_3-r_2r_3)e^{-2\gamma_2\ell_2}\right]}{\left(1-r_or_1e^{-2\gamma_1\ell_1}\right)\left(1-r_2r_3e^{-2\gamma_2\ell_2}\right) - r_ot_2r_3t_1e^{-2\gamma_1\ell_1}e^{-2\gamma_2\ell_2}} \tag{92}$$

$$(\bar{w}_1^-) = \frac{\bar{\mathcal{F}}}{i\omega(R_1+R_2)} \; \frac{e^{-\gamma_1\ell_1}\left[1 + \left(t_1r_3-r_2r_3e^{-2\gamma_2\ell_2}\right)\right]}{\left(1-r_or_1e^{-2\gamma_1\ell_1}\right)\left(1-r_2r_3e^{-2\gamma_2\ell_2}\right) - r_ot_2r_3t_1e^{-2\gamma_1\ell_1}e^{-2\gamma_2\ell_2}} \tag{93}$$

Eqns. (92) and (93) give the wave-mode coordinates $\bar{w}_1^+$ and $\bar{w}_1^-$ in terms of $\bar{\mathcal{F}}$. The physical components of the state vector at point 1 are given in terms of the wave-mode coordinates at point 1 by eqn. (51) as

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{F}_1 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ -iR_1\omega & iR_1\omega \end{bmatrix} \begin{Bmatrix} \bar{w}_1^+ \\ \bar{w}_1^- \end{Bmatrix} \tag{94}$$

Substitution of eqns. (92) and (93) into the second scalar equation of eqn. (94) gives

$$\bar{F}_1 = \bar{\mathscr{F}}\left[\frac{R_1}{R_1+R_2}\;\frac{(1-r_o)e^{-\gamma_1\ell_1}\left(1+(t_1r_3-r_2r_3)e^{-2\gamma_2\ell_2}\right)}{\left(1-r_or_1e^{-2\gamma_1\ell_1}\right)\left(1-r_2r_3e^{-2\gamma_2\ell_2}\right)-r_ot_2r_3t_1e^{-2\gamma_1\ell_1}e^{-2\gamma_2\ell_2}}\right] \tag{95}$$

Eqn. (95) gives the Fourier transform of $F_1(t)$ in terms of the Fourier transform of $\mathscr{F}(t)$. Thus, the quantity in the square brackets in eqn. (95) is the transfer function between the force $\mathscr{F}$ and the force F_1. As will be shown below, the transfer function in eqn. (95) can be written as an infinite series of constants multiplied by pure delays. It is noted in passing that the values of ω for which the denominator of the transfer function in eqn. (95) vanishes are the natural frequencies of the lattice of Fig. 5, and that there is an infinite number of such frequencies. Thus, the lattice of Fig. 5 is, in the language of linear system theory, an "infinite-order system."

In order to find $F_1(t)$, it is necessary first to know $\bar{\mathscr{F}}(\omega)$, and then to perform an inverse Fourier transform on eqn. (95). In this example it is assumed that

$$\mathscr{F}(t) = \mathscr{F}_o\delta(t) \tag{96}$$

where $\delta(t)$ is the Dirac delta function, so that [2]

$$\bar{\mathscr{F}}(\omega) = \mathscr{F}_o \tag{97}$$

Substitution of eqn. (97) into eqn. (95) gives

$$\bar{F}_1 = \mathscr{F}_o \left[\frac{R_1}{R_1+R_2} \; \frac{(1-r_o)\, e^{-\gamma_1 \ell_1} \left(1 + (t_1 r_3 - r_2 r_3) e^{-2\gamma_2 \ell_2}\right)}{\left(1-r_o r_1 e^{-2\gamma_1 \ell_1}\right)\left(1-r_2 r_3 e^{-2\gamma_2 \ell_2}\right) - r_o t_2 r_3 t_1 e^{-2\gamma_1 \ell_1} e^{-2\gamma_2 \ell_2}} \right] \tag{98}$$

which can be written as

$$\bar{F}_1 = \frac{\mathscr{F}_o \left(\frac{R_1}{R_1+R_2}\right)(1-r_o)\, \frac{r_3 t_1 e^{-2\gamma_1 \ell_1} e^{-2\gamma_2 \ell_2} + \left(1-r_2 r_3 e^{-2\gamma_2 \ell_2}\right) e^{-\gamma_1 \ell_1}}{\left(1 - r_o r_1 e^{-2\gamma_1 \ell_1}\right)\left(1 - r_2 r_3 e^{-2\gamma_2 \ell_2}\right)}}{1 - \frac{r_o t_2 r_3 t_1 e^{-2\gamma_1 \ell_1} e^{-2\gamma_2 \ell_2}}{\left(1-r_o r_1 e^{-2\gamma_1 \ell_1}\right)\left(1-r_2 r_3 e^{-2\gamma_2 \ell_2}\right)}} \tag{99}$$

By repeated use of the identity [12]

$$\left(\frac{1}{1-z}\right)^n = \frac{1}{(n-1)!} \sum_{m=0}^{\infty} (m+1)(m+2) \;\ldots\; (n+m-1) z^m \tag{100}$$

eqn. (99) can be written as [11]

$$\bar{F}_1 = \mathscr{F}_o\left(\frac{R_1}{R_1+R_2}\right)(1-r_o)$$

$$\cdot \left\{ \sum_{n=0}^{\infty} (r_o r_1)^n e^{-(2n+1)\gamma_1 \ell_1} \right.$$

$$+ \sum_{n=1}^{\infty} (r_o t_2 r_3 t_1)^n \left(\sum_{m=0}^{\infty} (r_o r_1)^m P(n+1,m) e^{-2n\gamma_1 \ell_1} e^{-(2m+1)\gamma_1 \ell_1} \right)$$

$$\cdot \left(\sum_{m=0}^{\infty} (r_2 r_3)^m P(n,m) e^{-2n\gamma_2 \ell_2} e^{-2m\gamma_2 \ell_2} \right)$$

$$+ \sum_{n=0}^{\infty} r_3 t_1 (r_o t_2 r_3 t_1)^n \left(\sum_{m=0}^{\infty} (r_o r_1)^m P(n+1,m) e^{-2n\gamma_1 \ell_1} e^{-(2m+1)\gamma_1 \ell_1} \right)$$

$$\left. \cdot \left(\sum_{m=0}^{\infty} (r_2 r_3)^m P(n+1,m) e^{-2n\gamma_2 \ell_2} e^{-2(m+1)\gamma_2 \ell_2} \right) \right\} \tag{101}$$

where

$$P(n,m) = \frac{(n+m-1)!}{(n-1)!m!} \tag{102}$$

Now, recalling the definitions of γ_1 and γ_2 given by eqns. (89) and (91), and the fact that the inverse Fourier transform of $e^{-i\omega\tau}$ is $\delta(t-\tau)$ [2], the inverse Fourier transform of eqn. (101) can be taken term by term to give

$$F_1(t) = \mathscr{F}_o \left(\frac{R_1}{R_1+R_2}\right)(1-r_o)$$

$$\cdot\ \delta(t) \Bigg\{ \sum_{n=0}^{\infty} (r_o r_1)^n \lambda((2n+1)\tau)$$

$$+ \sum_{n=1}^{\infty} (r_o t_2 r_3 t_1)^n \left(\sum_{m=0}^{\infty} (r_o r_1)^m P(n+1,m) \lambda(2n\tau_1) \lambda((2m+1)\tau_1) \right)$$

$$\cdot \left(\sum_{m=0}^{\infty} (r_2 r_3)^m P(n,m) \lambda(2n\tau_2) \lambda(2m\tau_2) \right)$$

$$+ \sum_{n=0}^{\infty} r_3 t_1 (r_o t_2 r_3 t_1)^n \left(\sum_{m=0}^{\infty} (r_o r_1)^m P(n+1,m) \lambda(2n\tau_1) \lambda((2m+1)\tau_1) \right)$$

$$\cdot \left(\sum_{m=0}^{\infty} (r_2 r_3)^m P(n+1,m) \lambda(2n\tau_2) \lambda(2(m+1)\tau_2) \right) \Bigg\} \tag{103}$$

where

$$\tau_1 = \ell_1 \sqrt{\frac{\rho_1}{E_1}} \tag{104}$$

$$\tau_2 = \ell_2 \sqrt{\frac{\rho_2}{E_2}} \tag{105}$$

and $\lambda(\tau)$ is a time-shift factor defined by

$$f(t)\lambda(\tau) = f(t-\tau) \tag{106}$$

Writing out the first few terms of eqn. (130) gives

$$F_1(t) = \mathscr{F}_o\left(\frac{R_1}{R_1+R_2}\right)(1-r_o)$$

$$\cdot \Bigg\{ \delta(t)\left[\lambda(\tau_1) + r_o r_1 \lambda(3\tau_1) + (r_o r_1)^2 \lambda(5\tau_1) + \ldots\right]$$

$$+ \delta(t) r_3 t_1 \left[\lambda(\tau_1) + r_o r_1 \lambda(3\tau_1) + (r_o r_1)^2 \lambda(5\tau_1) + \ldots\right]$$

$$\cdot \left[\lambda(2\tau_2) + r_2 r_3 \lambda(4\tau_2) + (r_2 r_3)^2 \lambda(6\tau_2) + \ldots\right]$$

$$+ \delta(t) r_o t_2 r_3 t_1 \left[\lambda(3\tau_1) + 2 r_o r_1 \lambda(5\tau_1) + 3(r_o r_1)^2 \lambda(7\tau_1) + \ldots\right]$$

$$\cdot \left[\lambda(2\tau_2) + r_2 r_3 \lambda(4\tau_2) + (r_2 r_3)^2 \lambda(6\tau_2) + \ldots\right]$$

$$+ \delta(t) r_3 t_1 (r_o t_2 r_3 t_1) \left[\lambda(3\tau_1) + 2 r_o r_1 \lambda(5\tau_1) + 3(r_o r_1)^2 \lambda(7\tau_1) + \ldots\right]$$

$$\cdot \left[\lambda(4\tau_2) + 2 r_2 r_3 \lambda(6\tau_2) + 3(r_2 r_3)^2 \lambda(8\tau_2) + \ldots\right]$$

$$+ \delta(t) (r_o t_2 r_3 t_1)^2 \left[\lambda(5\tau_1) + 3 r_o r_1 \lambda(7\tau_1) + 6(r_o r_1)^2 \lambda(9\tau_1) + \ldots\right]$$

$$\cdot \left[\lambda(4\tau_2) + 2 r_2 r_3 \lambda(6\tau_2) + 3(r_2 r_3)^2 \lambda(8\tau_2) + \ldots\right]$$

$$+ \delta(t) r_3 t_1 (r_o t_2 r_3 t_1)^2 \left[\lambda(5\tau_1) + 3 r_o r_1 \lambda(7\tau_1) + 6(r_o r_1)^2 \lambda(9\tau_1) + \ldots\right]$$

$$\cdot \left[\lambda(6\tau_2) + 3 r_2 r_3 \lambda(8\tau_2) + 6(r_2 r_3)^2 \lambda(10\tau_2) + \ldots\right]$$

$$+ \ldots \Bigg\} \qquad (107)$$

Eqn. (103), which is an infinite series of impulses scaled in amplitude by reflection and transmission coefficients and delayed by multiples of the times τ_1 and τ_2, is the desired solution for $F_1(t)$.

Each of the impulses in eqn. (107) can be interpreted physically by considering the propagation of the initial impulse applied to the joint. The initial impulse $\mathscr{F}$ causes an impulse to propagate to the left towards point 1, and also causes an impulse to propagate to the right towards point 4. After a time delay τ_1, the impulse which initially propagates to the left arrives at point 1, where it is reflected and scaled in amplitude by the reflection coefficient r_o. The reflected impulse arrives at the joint after an additional time delay τ_1, and at the joint it is partially reflected back towards point 1, and partially transmitted towards point 4. The reflected impulse is scaled in amplitude by the reflection coefficient r_1, and the transmitted impulse is scaled in amplitude by the transmission coefficient t_2. The impulse which initially propagates from the joint to the right arrives, after a time delay τ_2, at point 4, where it is reflected and scaled in amplitude by the reflection coefficient r_3. The reflected impulse arrives at the joint after an additional time delay τ_2, and at the joint it is partially reflected back towards point 4, and partially transmitted towards point 1. The reflected impulse is scaled in amplitude by the reflection coefficient r_2, and the transmitted impulse is scaled in amplitude by the transmission coefficient t_1. This process of propagation, reflection, and transmission continues indefinitely. Since the propagation constants and the scattering matrices for the lattice of Fig. 5 are nondispersive, the initial impulse applied to the joint maintains its identity as an impulse as it propagates and as it is reflected and transmitted.

The first infinite series in eqn. (107) represents those impulses which initially propagate from the joint to the left, and then remain in rod 1 due to successive reflections at point 1 and at the joint. The first product of two infinite series in eqn. (107) represents those impulses which initially propagate from the joint to the right, and then, after an arbitrary number of reflections in rod 2, are transmitted through the joint into rod 1, and remain thereafter in rod 1. The second product of two infinite series in eqn. (107) represents those impulses which initially propagate from the joint to the left, reflect an arbitrary number of times in rod 1, are transmitted through the joint into rod 2, reflect an arbitrary number of times in rod 2, are transmitted through the joint into rod 1, and remain thereafter in rod 1. The third product of two infinite series in eqn. (107) represent those impulses which initially propagate from the joint to the right, reflect an arbitrary number of times in rod 2, are transmitted through the joint into rod 1, reflect an arbitrary number of times in rod 1, are transmitted through the joint into rod 2, reflect an arbitrary number of times in rod 2, are transmitted through the joint into rod 1, and remain thereafter in rod 1. The remaining terms in eqn. (103) can be interpreted in a similar manner.

The response $F_1(t)$ given by eqn. (103) is obtained here by using wave-mode coordinates and scattering matrices in the frequency domain to obtain an expression $\tilde{F}_1(\omega)$, and then performing an inverse Fourier transform to obtain $F_1(t)$. The response $F_1(t)$ can also be obtained by a direct consideration of propagation, reflection, and transmission in the time domain. Such an approach is taken in [13]. The response $F_1(t)$ given here agrees exactly with the response which is obtained using the methods given in [13]. In fact, the numerical coefficients which appear in the various infinite series of eqn. (103), and which occur here naturally as a part of the process of inverse Fourier transformation, account for the existence of "equivalent paths" from the input location (the joint) to the response location (point 1). The concept of equivalent paths in a lattice structure is discussed thoroughly in Appendix A of [13], and some sketches of equivalent paths in the lattice of Fig. 5 are given in [11]. The advantage of the approach taken here, which uses wave-mode coordinates and scattering matrices, is that this approach may be extended, using the same formal procedures, to the analysis of two and three-dimensional lattice structures.

CONCLUSIONS

The examples given in the previous section illustrate, in principle, all the steps required for an analysis of wave propagation in an arbitrary lattice structure. The basic ingredients of such an analysis are a wave-mode matrix $\underline{Y}(\omega)$ and a wave-mode propagation matrix $\underline{W}(\omega)$ for each member of the lattice and a scattering matrix $\underline{S}(\omega)$ and a wave-mode generation matrix $\underline{G}(\omega)$ for each joint in the lattice. A wave-mode propagation relationship such as eqn. (7) applied to each member and a scattering relationship such as eqn. (25) applied to each joint provide a set of equations which can be solved for the wave-mode vector $\underline{\bar{w}}$ at any point in the lattice. The state vector $\underline{\bar{z}}$ at any point in the lattice can then be obtained through the use of the wave-mode transformation given by eqn. (5). Finally, individual physical components of the state vector $\underline{z}$ may be obtained through the process of inverse Fourier transformation.

The major difficulty in the procedure just described is the evaluation of the inverse Fourier transform. Efficient numerical techniques for evaluating complicated inverse Fourier transforms are necessary if the methods described in this paper are to be used successfully in the analysis of large and complex lattice structures.

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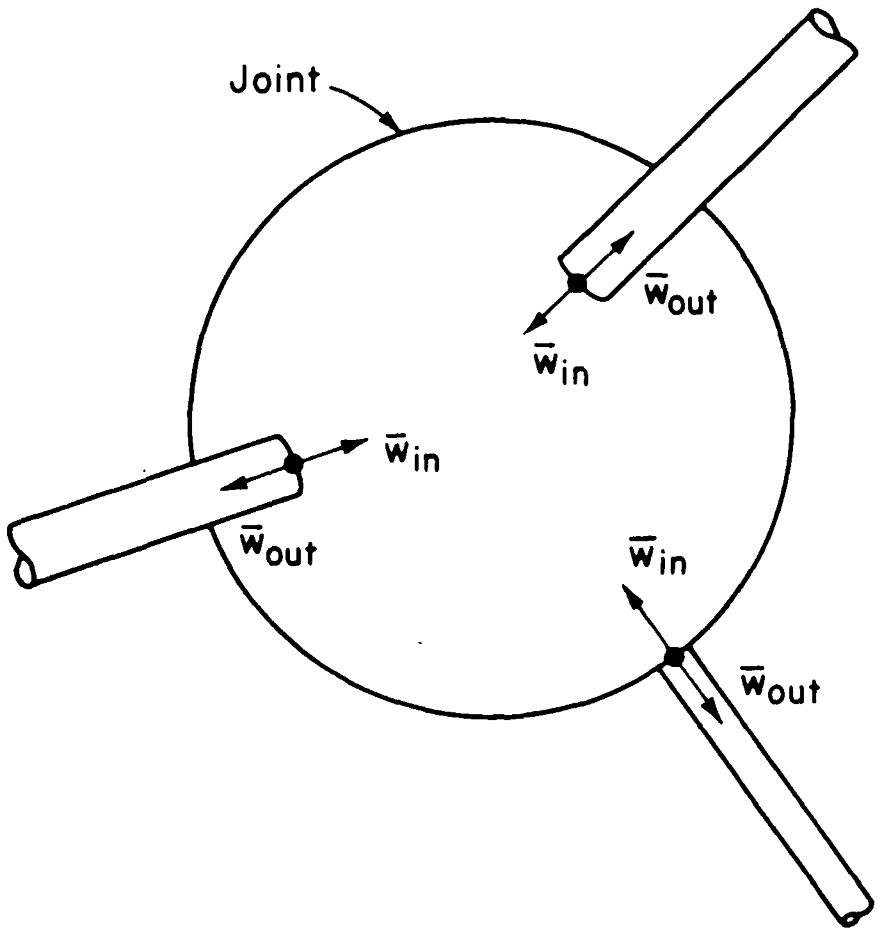

Fig. 1 Wave-mode coordinates entering and leaving a generic joint.

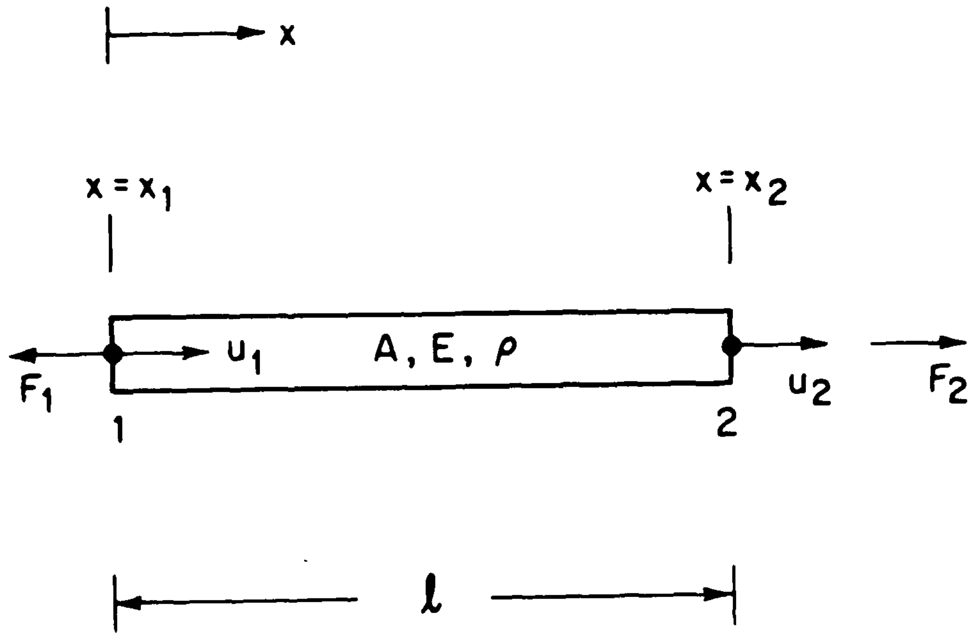

Fig. 2 Elastic longitudinal rod.

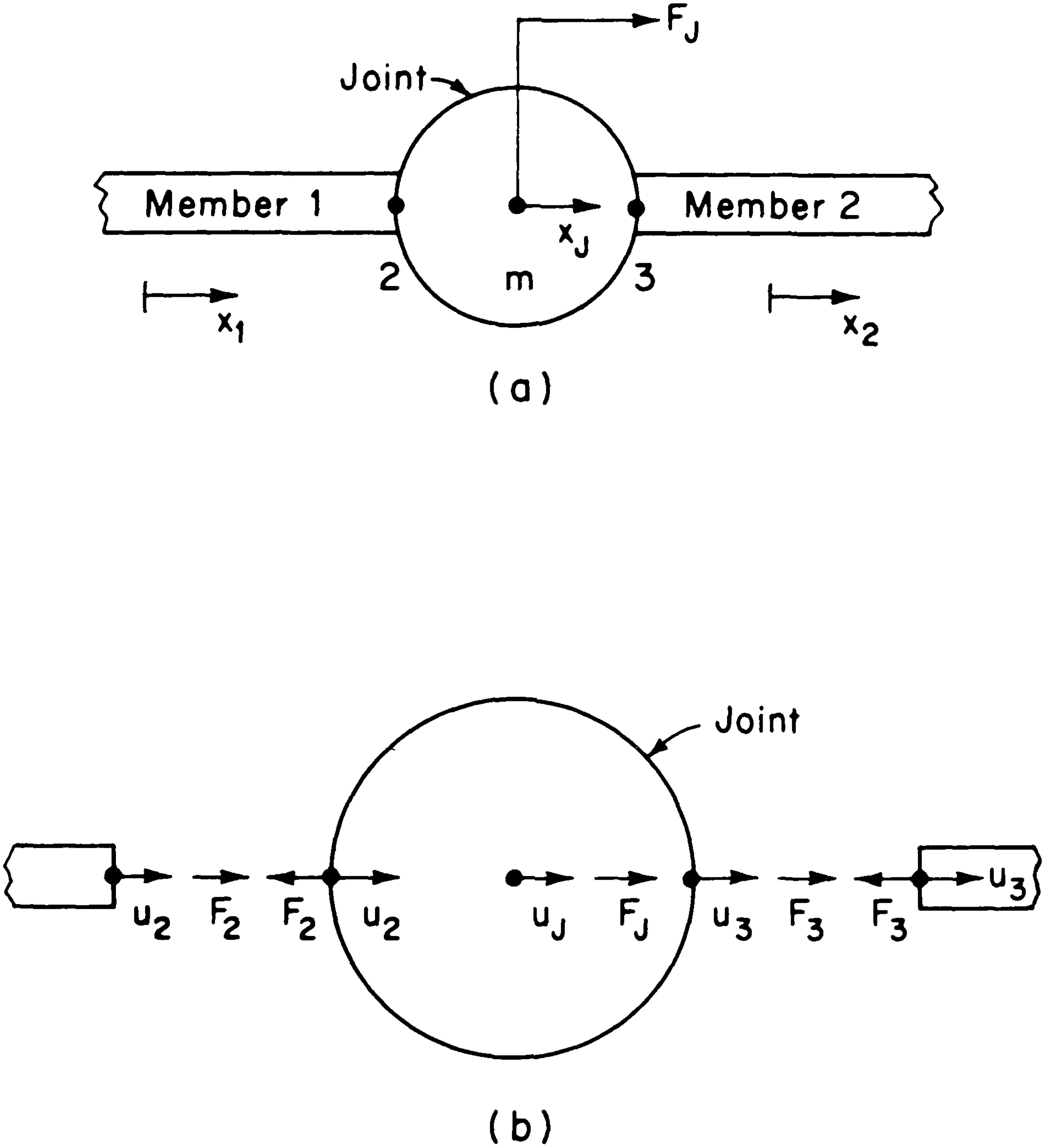

Fig. 3 Rigid one-dimensional joint.

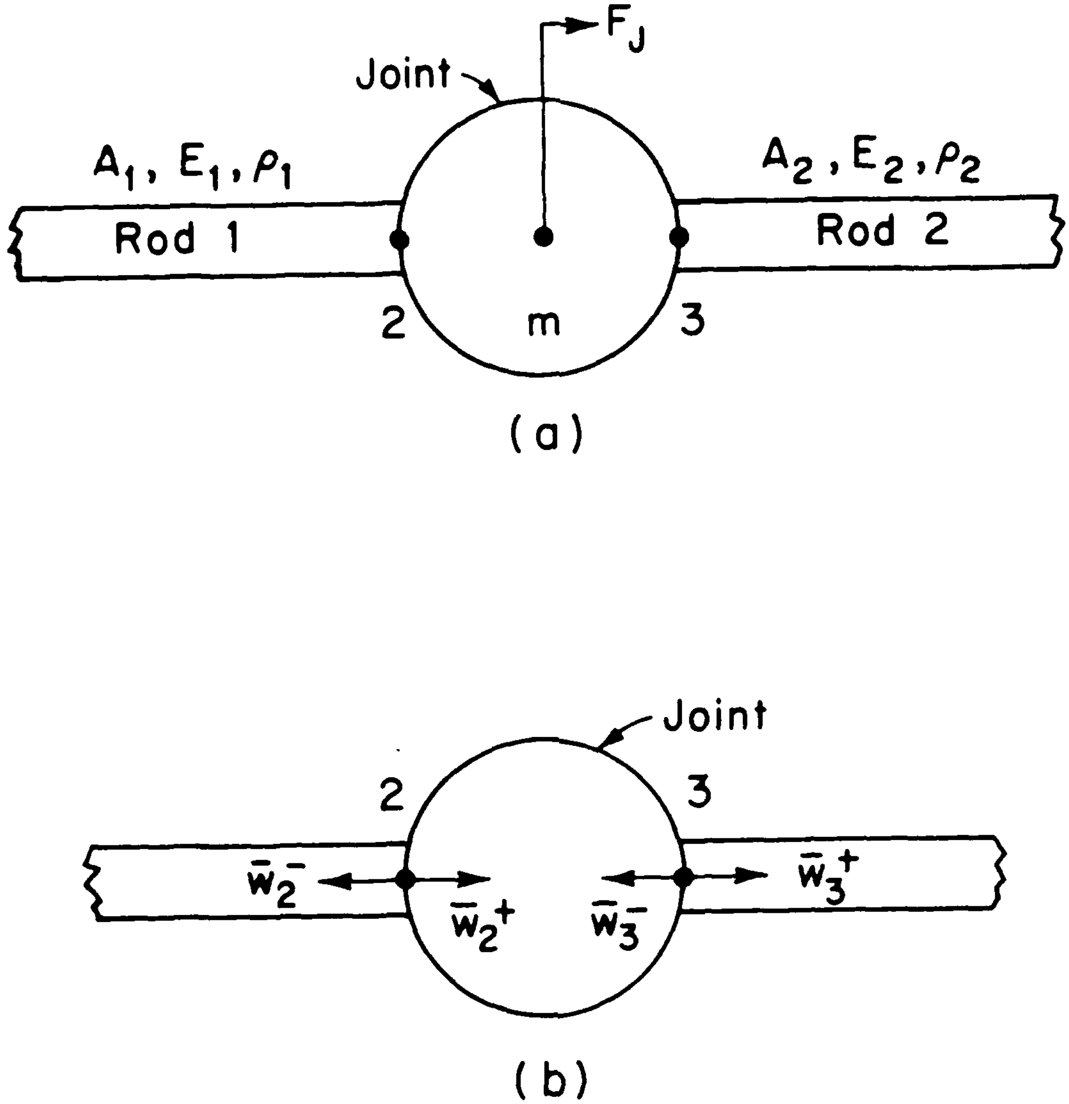

Fig. 4 Two elastic longitudinal rods attached to a rigid one-dimensional joint.

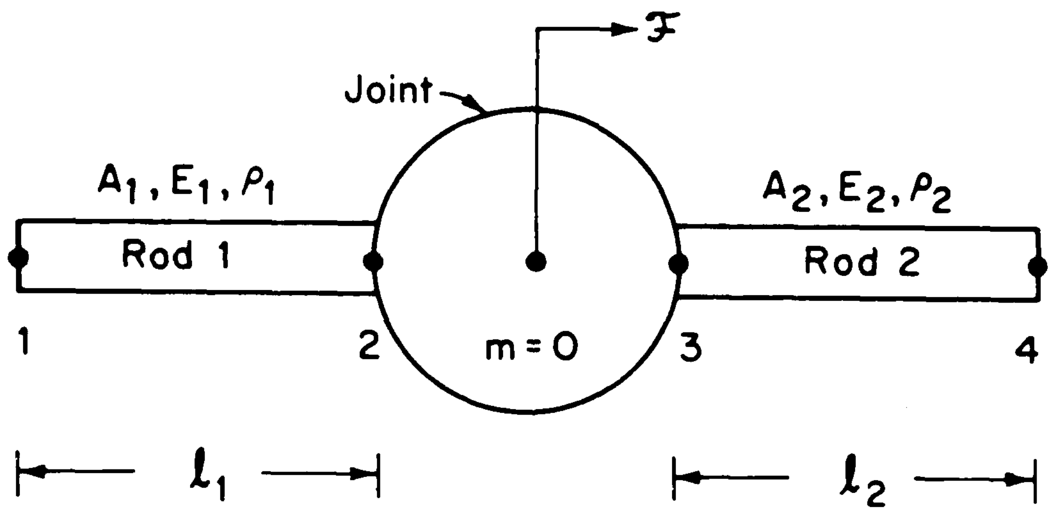

Fig. 5 One-dimensional lattice structure.

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